

# More on Asymptotically Anti-de Sitter Spaces in Topologically Massive Gravity

Marc Henneaux<sup>1,2</sup>, Cristián Martínez<sup>1,3</sup>, Ricardo Troncoso<sup>1,3\*</sup>

<sup>1</sup>*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia,*

<sup>2</sup>*Physique théorique et mathématique,*

*Université Libre de Bruxelles and International Solvay Institutes,*

*ULB Campus Plaine C.P.231, B-1050 Bruxelles, Belgium, and*

<sup>3</sup>*Centro de Ingeniería de la Innovación del CECS (CIN), Valdivia, Chile.*

## Abstract

Recently, the asymptotic behaviour of three-dimensional anti-de Sitter gravity with a topological mass term was investigated. Boundary conditions were given that were asymptotically invariant under the two-dimensional conformal group and that included a fall-off of the metric sufficiently slow to consistently allow pp-wave type of solutions. Now, pp-waves can have two different chiralities. Above the chiral point and at the chiral point, however, only one chirality can be considered, namely the chirality that has the milder behaviour at infinity. The other chirality blows up faster than AdS and does not define an asymptotically AdS spacetime. By contrast, both chiralities are subdominant with respect to the asymptotic behaviour of AdS spacetime below the chiral point. Nevertheless, the boundary conditions given in the earlier treatment only included one of the two chiralities (which could be either one) at a time. We investigate in this paper whether one can generalize these boundary conditions in order to consider simultaneously both chiralities below the chiral point. We show that this is not possible if one wants to keep the two-dimensional conformal group as asymptotic symmetry group. Hence, the boundary conditions given in the earlier treatment appear to be the best possible ones compatible with conformal symmetry. In the course of our investigations, we provide general formulas controlling the asymptotic charges for all values of the topological mass (not just below the chiral point).

**Keywords:** Three-dimensional gravity, asymptotic conditions.

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\*Electronic address: henneaux@ulb.ac.be, martinez@cecs.cl, troncoso@cecs.cl

## I. INTRODUCTION

Topologically massive gravity in three dimensions with a negative cosmological constant [1–3], described by the action

$$I[e] = 2 \int \left[ e^a \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c \right) + \frac{1}{6} \frac{1}{\ell^2} \epsilon_{abc} e^a e^b e^c \right] + \frac{1}{\mu} \int \left[ \omega^a \left( d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^b \omega^c \right) \right] \quad (\text{I.1})$$

admits a rich variety of non trivial solutions (see [4, 5] for recent reviews and new solutions)<sup>1</sup>. Among these, pp-waves [6, 7] are particularly interesting as they preserve supersymmetry [8]. For a given value of the topological mass parameter  $\mu\ell \neq \pm 1$ , there are two chiralities, described by

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - r^2 dx^+ dx^- + F(x^-) r^{1-\mu\ell} (dx^-)^2 \quad (\text{I.2})$$

(negative chirality) and

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - r^2 dx^+ dx^- + G(x^+) r^{1+\mu\ell} (dx^+)^2 \quad (\text{I.3})$$

(positive chirality), where  $F$  and  $G$  are arbitrary functions. We assume the basis  $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial x^+}, \frac{\partial}{\partial x^-}\}$  to have positive orientation.

For a standard pp-wave, the coordinates  $t$  and  $\phi$ , related to  $x^+$  and  $x^-$  according to  $x^\pm = \frac{t}{\ell} \pm \phi$ , are assumed to be of infinite range,  $-\infty < t < +\infty$ ,  $-\infty < \phi < +\infty$ . However, one may clearly regard  $\phi$  as an angle,  $0 \leq \phi \leq 2\pi$  without changing the fact that (I.2) and (I.3) are solutions. This is what we shall do here since we want to study spaces which are asymptotical to anti-de Sitter space whose metric reads in standard static coordinates

$$d\bar{s}^2 = \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 - \frac{\ell^2}{4} (dx^{+2} + dx^{-2}) - \left(\frac{\ell^2}{2} + r^2\right) dx^+ dx^-,$$

where  $\phi$  is an angle.

In the chiral case  $\mu\ell = \pm 1$ , which has attracted much attention recently [9] following the lead of [10], the pp-wave solutions acquire a logarithmic behaviour in  $r$ . For  $\mu\ell = 1$ , they read explicitly

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - r^2 dx^+ dx^- + F(x^-) \log r (dx^-)^2 \quad (\text{I.4})$$

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<sup>1</sup> Our conventions are as follows:  $\mu \neq 0$  is the mass parameter,  $\ell$  is the AdS radius,  $\epsilon_{012} = 1$  (so  $\epsilon^{012} = -1$ ) and we have set the gravitational coupling constant  $16\pi G = 1$ .

(negative chirality) and

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - r^2 dx^+ dx^- + G(x^+) r^2 \log r (dx^+)^2 \quad (\text{I.5})$$

(positive chirality), where  $F$  and  $G$  are again arbitrary functions. The solutions for  $\mu\ell = -1$  are simply obtained by exchanging  $x^+$  and  $x^-$ .

When  $|\mu\ell| \geq 1$ , only one of the solutions (I.2) and (I.3) (or (I.4) and (I.5)) asymptotically matches AdS in the given coordinate system, namely the negative chirality solution if  $\mu\ell \geq 1$  or the positive chirality solution if  $\mu\ell \leq -1$ . Indeed, it is only for that solution that the term containing  $F$  or  $G$  is subdominant with respect to the asymptotic behaviour  $\sim r^2$  of the angular part of the AdS metric. For the other solution the  $g_{++}$  ( $g_{--}$ ) component goes to infinity faster than  $r^2$ . When  $|\mu\ell| < 1$ , however, both solutions are such that  $g_{++}$  and  $g_{--}$  blow up at infinity more slowly than  $r^2$ .

In [11], boundary conditions are devised satisfying the consistency requirements spelled out in [12]:

- They are invariant under the anti-de Sitter group.
- They decay sufficiently slowly to the exact anti-de Sitter metric at infinity so as to contain the “asymptotically anti-de Sitter” solutions of the theory of physical interest (see below for precise statements on this point).
- But at the same time, the fall-off is sufficiently fast so as to yield finite charges.

The boundary conditions of [11] generalize those of [13] by accommodating more solutions. Not only do they contain the BTZ black holes [14, 15], as [13] does, but they also include in addition the slower asymptotic behaviour of the above pp-wave metrics. More precisely, they enable one to turn on one chirality, which is the chirality that does not blow up faster than AdS when  $|\mu\ell| \geq 1$ , and which can be any of the two chiralities when  $|\mu\ell| < 1$ .

In that latter case, however, the boundary conditions of [11] only allow to switch on one chirality at a time. This appears to be sufficient to accommodate the known exact solutions, which do not involve both chiralities. It is nevertheless somewhat disturbing since each chirality is then individually compatible with the asymptotic anti-de Sitter symmetry. One might in principle, by integrating from infinity, construct solutions where both chiralities are present.

It is therefore natural to ask whether one can extend the boundary conditions of [11] so as to allow simultaneously both chiralities when  $|\mu\ell| < 1$ .

The situation is somewhat similar to what happens when a massive scalar field is present, which has been extensively studied in [16–20]. When the mass  $m$  is in the range  $m_{BF}^2 < m^2 < m_{BF}^2 + 1/l^2$  (where  $m_{BF}$  is the Breitenlohner-Freedman bound [21]), there are two possible admissible behaviours (two “branches”) for the scalar field. One may devise boundary conditions which exclude one of the two branches. But one can also consider more general boundary conditions where both branches are switched on while preserving the asymptotically anti-de Sitter group.

We show in this paper that similarly a more general asymptotic treatment of topologically massive gravity exists when  $|\mu\ell| < 1$ . This more general treatment allows both chiralities. However, contrary to what happens in the scalar field case, asymptotic anti-de Sitter invariance is lost when both chiralities are simultaneously switched on<sup>2</sup>. It appears therefore that the boundary conditions of [11] are the most general boundary conditions compatible with full anti-de Sitter symmetry.

What happens can be roughly understood as follows (the precise analysis is given below). The two chiralities are conjugate, as are the two branches of the scalar field. Hence, in the variational principle, one cannot vary them independently but one must fix a relation between them, just as one cannot leave both the  $q$ ’s and the  $p$ ’s free in the Hamiltonian variational principle. While in the scalar case, the relation between both branches can be chosen to be anti-de Sitter invariant without killing one branch, this is not the case here. The only anti-de Sitter invariant relations are obtained by setting one of the two chiralities to zero.

Our paper is organized as follows. In the next section, we give the boundary conditions that incorporate both chiralities. When one sets one chirality to zero, one recovers the boundary conditions of [11]. We then show that these boundary conditions, without additional restrictions, are invariant under the full conformal group at infinity. However, the need to have well-defined charges, whose variations are not only finite but also integrable, forces one to impose a relation on the two chiralities. In Section III, we prove that there is no such relation that preserves the conformal symmetry at infinity, except setting one of

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<sup>2</sup> The results of this analysis were announced (but not proved) in [11].

the two chiralities to zero. The best that one can achieve otherwise is  $\text{Virasoro} \times R$ .

We collect in the appendices the technical tools necessary for handling the surface integrals. This gives us the opportunity to provide the detailed formulas promised in [11] (a special appendix is in particular reserved to the chiral point).

## II. BOUNDARY CONDITIONS FOR TMG WITH $|\mu\ell| < 1$

### A. Asymptotic Conditions

We tentatively take as boundary conditions

$$\begin{aligned}
\Delta g_{rr} &= f_{rr} r^{-4} + \dots \\
\Delta g_{r+} &= h_{r+} r^{-2+\mu\ell} + f_{r+} r^{-3} + \dots \\
\Delta g_{r-} &= h_{r-} r^{-2-\mu\ell} + f_{r-} r^{-3} + \dots \\
\Delta g_{++} &= h_{++} r^{1+\mu\ell} + f_{++} + \dots \\
\Delta g_{+-} &= f_{+-} + \dots \\
\Delta g_{--} &= h_{--} r^{1-\mu\ell} + f_{--} + \dots
\end{aligned} \tag{II.1}$$

where  $f_{\mu\nu}$  and  $h_{\mu\nu}$  depend only on  $x^+$  and  $x^-$  and not on  $r$ . We use the convention that the  $f$ -terms are the standard deviations from AdS considered in [13], while the  $h$ -terms represent the relaxed terms that need to be included in order to accommodate the solutions of the topologically massive theory with slower fall-off.

When only one chirality is included, one recovers the boundary conditions of [11]. This is how (II.1) was arrived at, by simply superposing the boundary conditions for the individual chiralities considered in that reference.

We shall see below that these boundary conditions need to be strengthened. Integrability of the charges forces indeed a relationship between  $h_{++}$  and  $h_{--}$ .

## B. Asymptotic Symmetry

One easily verifies that the asymptotic conditions without the extra relation  $h_{++} = h_{--}$  are invariant under diffeomorphisms that behave at infinity as

$$\begin{aligned}\eta^+ &= T^+ + \frac{l^2}{2r^2} \partial_-^2 T^- + \dots \\ \eta^- &= T^- + \frac{l^2}{2r^2} \partial_+^2 T^+ + \dots \\ \eta^r &= -\frac{r}{2} (\partial_+ T^+ + \partial_- T^-) + \dots\end{aligned}\tag{II.2}$$

where  $T^\pm = T^\pm(x^\pm)$ . The  $\dots$  terms are of lowest order and do not contribute to the surface integrals. Hence, the tentative boundary conditions (II.1) are invariant under the full conformal group in two dimensions, generated by  $T^+(x^+)$  and  $T^-(x^-)$ .

## III. SURFACE TERMS

### A. Form of surface terms

The conserved charges are computed within the canonical formalism, “à la Regge-Teitelboim” [22]. The Hamiltonian formalism for topologically massive gravity is reviewed in appendix A.

The searched-for charges that generate the diffeomorphisms (II.2) must take the form [22]

$$H[\eta] = \text{“Bulk piece”} + Q_+[T^+] + Q_-[T^-],\tag{III.1}$$

where the bulk piece is a linear combination of the constraints with coefficients involving  $\eta^+, \eta^-, \eta^r$  which has been explicitly worked out in [23] and which are given in the appendices, and where  $Q_+[T^+]$  and  $Q_-[T^-]$  are surface integrals at infinity that involve only the asymptotic form of the vector field  $\eta^+, \eta^-, \eta^r$ . On-shell, the bulk piece vanishes and  $H[\eta]$  reduces to  $Q_+[T^+] + Q_-[T^-]$ . As explained in [22], the variation of the surface integrals at infinity must cancel, under the given boundary conditions, the surface terms that one picks up upon integrations by parts in the bulk term. So, what is determined by the formalism are the variations  $\delta Q_\pm$  of the surface terms.

A crucial consistency requirement on the boundary conditions is that these variations  $\delta Q_\pm$  should be integrable, i.e., should define exact forms in field space. If they are not, one

must strengthen the boundary conditions so as to fulfill this requirement.

The variation of the surface integrals giving the conserved charges under the boundary conditions (II.1) are shown in appendix B to be equal to

$$\delta Q_{\pm} = \left(1 \pm \frac{1}{\mu l}\right) \delta Q_{\pm}^0 + \delta Q_{\pm}^{nl} , \quad (\text{III.2})$$

where

$$\delta Q_{\pm}^0 := \frac{2}{l} \int T^{\pm} \delta f_{\pm\pm} d\phi , \quad (\text{III.3})$$

is the standard contribution that one finds with the asymptotic behavior of [13], and

$$\delta Q_{\pm}^{nl} = \pm \frac{1}{2l} (\mu^2 l^2 - 1) \int T^{\pm} \left[ \left( \frac{3}{\mu l} \mp 1 \right) h_{\pm\pm} \delta h_{\mp\mp} + \left( \frac{1}{\mu l} \mp 1 \right) h_{\mp\mp} \delta h_{\pm\pm} \right] d\phi , \quad (\text{III.4})$$

is the additional nonlinear contribution coming from the relaxed terms containing  $h_{\mu\nu}$ .

Now, while the standard contribution is integrable, the additional term is not if we assume no restriction on the space of the  $h_{\mu\nu}$ 's. Indeed, one gets that the second variation of the additional piece is given by

$$\delta^2 Q_{\pm}^{nl} \sim \delta h_{++} \wedge \delta h_{--} ,$$

which is non-zero unless we assume a functional dependence  $h_{++} = h_{++}(h_{--})$  (or  $h_{--} = h_{--}(h_{++})$ ). As mentioned in the introduction, the same situation is encountered when a scalar field is coupled to gravity. There are two branches for the scalar field, behaving asymptotically as

$$\frac{a_{\pm}}{r^{\lambda_{\pm}}} .$$

Integrability of the anti-de Sitter charges forces a functional relation between  $a_{+}$  and  $a_{-}$ .

## B. Constraints from asymptotic conformal invariance

The functional relation must be chosen to be invariant under the asymptotic symmetry. This is a non-trivial requirement because  $h_{++}$  and  $h_{--}$  transform differently.

Under the action of the Virasoro symmetry, one obtains

$$\delta_{\eta} h_{++} = \frac{1}{2} \left[ (3 - \mu l) \partial_{+} T^{+} - (\mu l + 1) \partial_{-} T^{-} \right] h_{++} + T^{-} \partial_{-} h_{++} + T^{+} \partial_{+} h_{++} , \quad (\text{III.5})$$

$$\delta_{\eta} h_{--} = \frac{1}{2} \left[ (\mu l - 1) \partial_{+} T^{+} + (3 + \mu l) \partial_{-} T^{-} \right] h_{--} + T^{-} \partial_{-} h_{--} + T^{+} \partial_{+} h_{--} . \quad (\text{III.6})$$

The searched-for relation  $h_{++} = h_{++}(h_{--})$  must be consistent with these equations, i.e., one must have

$$\delta_\eta h_{++} = \frac{\delta h_{++}}{\delta h_{--}} \delta_\eta h_{--} . \quad (\text{III.7})$$

This has to be true for each Virasoro copy.

Considering first the right copy (generated by  $T^+$ ), one gets

$$\left( (3 - \mu l) h_{++} + (1 - \mu l) \frac{\delta h_{++}}{\delta h_{--}} h_{--} \right) \partial_+ T^+ = 0 , \quad (\text{III.8})$$

which integrates into

$$h_{++} = a_0^+ (h_{--})^{\frac{3-\mu l}{\mu l-1}} , \quad (\text{III.9})$$

Similarly, one finds for the left copy (generated by  $T^-$ )

$$\left( (1 + \mu l) h_{++} + (3 + \mu l) \frac{\delta h_{++}}{\delta h_{--}} h_{--} \right) \partial_- T^- = 0 , \quad (\text{III.10})$$

which integrates into

$$h_{++} = a_0^- (h_{--})^{-\frac{1+\mu l}{\mu l+3}} . \quad (\text{III.11})$$

In these equations,  $a_0^\pm$  are integration constants. As was to be expected, conditions (III.9) and (III.11) are interchanged under  $\mu \longleftrightarrow -\mu$  and  $x^+ \longleftrightarrow x^-$ .

### C. Conformal symmetry at infinity

As there is no value of  $\mu$  for which the powers of  $h_{--}$  in the equations (III.9) and (III.11) are equal, the full conformal invariance at infinity is absent, unless one sets the integration constants equal to zero or infinity; i.e., only when one chirality is switched off. This was the case considered in [11]. (Note that when  $h_{\pm\pm}$  is switched off,  $h_{r_\pm}$  can be gauged away.) This is in sharp contrast with what is found in the scalar field case where the functional relation  $a_+(a_-)$  can be chosen to be invariant without removing one of the two branches.

Now, when one of the chiralities is set to zero, the extra nonlinear contribution (III.4) vanishes and the charges reduce to the ones obtained for the standard asymptotic conditions, namely,

$$Q_\pm[T^\pm] = \frac{2}{l} \left( 1 \pm \frac{1}{\mu l} \right) \int T^\pm f_{\pm\pm} d\phi. \quad (\text{III.12})$$

Therefore, somewhat unexpectedly, the charges acquire no correction due to the relaxed terms in the asymptotic expansion. Since these terms cannot be gauged away, they can



be viewed as “massive graviton hair”. Note in particular that the charges above for the pp-waves (with  $\phi$  identified) are all zero.

Once the generators of the asymptotic symmetries have been found, one can compute their algebra applying the general theorems of [24]. These guarantee that the algebra is the algebra of the conformal group with possible central charges. The central charges are easily found from the inhomogeneous transformation terms (under the conformal symmetry) in the variations of the functions  $f_{\pm\pm}$  that determine the charges. One finds,

$$\delta_\eta f_{++} = \dots - l^2 (\partial_+ T^+ + \partial_+^3 T^+)/2 , \quad (\text{III.13})$$

$$\delta_\eta f_{--} = \dots - l^2 (\partial_- T^- + \partial_-^3 T^-)/2 , \quad (\text{III.14})$$

where  $\dots$  stand for the homogeneous terms. One can thus infer that  $Q_+(T^+)$  and  $Q_-(T^-)$  commute with each other and each fulfills the Virasoro algebra with central charges

$$c_\pm = \left(1 \pm \frac{1}{\mu l}\right) c , \quad (\text{III.15})$$

where  $c$  is the central charge of [13],

$$c = \frac{3l}{2G} .$$

#### D. Two chiralities present

If one does not insist on conformal invariance at infinity, one can switch on simultaneously both chiralities. If the relation between  $h_{++}$  and  $h_{--}$  is taken to be one of the two relations above, one preserves one chiral copy of the Virasoro algebra. From the other chiral copy, only the zero mode survives (the conditions (III.8), (III.10) which relate  $h_{++}$  to  $h_{--}$  are trivially satisfied when  $T^\pm$  are constants). The asymptotic symmetry is then  $\text{Virasoro} \times R$ .

It is easy to integrate the charges. Let us assume first that we adopt the relation (III.9) between  $h_{++}$  and  $h_{--}$ . Then, from Eq. (III.2) one verifies that  $\delta Q_+^{nl} = 0$  and then

$$Q_+(T^+) = \frac{2}{l} \left(1 + \frac{1}{\mu l}\right) \int T^+ f_{++} d\phi , \quad (\text{III.16})$$

for the Virasoro algebra with central charge

$$c_+ = \left(1 + \frac{1}{\mu l}\right) c , \quad (\text{III.17})$$

and

$$Q_-[T^-] = \frac{2}{l} \left(1 - \frac{1}{\mu l}\right) \int T^- \left[ f_{--} - (1 + \mu l) a_0^+ h_{--}^{\frac{2}{\mu l - 1}} \right] d\phi , \quad (\text{III.18})$$

for the generator of the zero mode ( $\partial_- T^- = 0$ ) (See appendices B and D).

Analogously, if one considers the relation (III.11) between  $h_{++}$  and  $h_{--}$ , one obtains

$$Q_-(T^-) = \frac{2}{l} \left(1 - \frac{1}{\mu l}\right) \int T^- f_{--} d\phi , \quad (\text{III.19})$$

for the Virasoro algebra with central charge

$$c_- = \left(1 - \frac{1}{\mu l}\right) c , \quad (\text{III.20})$$

and

$$Q_+(T^+) = \frac{2}{l} \left(1 + \frac{1}{\mu l}\right) \int T^+ \left[ f_{++} - (1 - \mu l) a_0^- (h_{--})^{\frac{2}{\mu l + 3}} \right] d\phi , \quad (\text{III.21})$$

for the generator of the zero mode ( $\partial_+ T^+ = 0$ ). Naturally, the expressions for the charges and the central extension associated to (III.11) coincide with the ones for (III.9) making  $\mu \longleftrightarrow -\mu$  and  $x^+ \longleftrightarrow x^-$ .

One can also break further the Virasoro symmetry through arbitrary boundary conditions. For a generic relation between  $h_{++}$  and  $h_{--}$  of the form

$$h_{++} = w'(h_{--}) , \quad (\text{III.22})$$

the conditions (III.8), (III.10) are fulfilled only for the zero modes of both copies of the Virasoro symmetry (i.e., when  $T^+$  and  $T^-$  are constants). Hence, the asymptotic symmetry is broken down to  $R \times U(1)$ , and the charges acquire the form

$$Q_+[T^+] = \frac{2}{l} \left(1 + \frac{1}{\mu l}\right) \int T^+ \left[ f_{++} + \frac{1}{4}(\mu l - 1) (2w - (\mu l - 1)w'h_{--}) \right] d\phi , \quad (\text{III.23})$$

$$Q_-[T^-] = \frac{2}{l} \left(1 - \frac{1}{\mu l}\right) \int T^- \left[ f_{--} + \frac{1}{4}(\mu l + 1) (2w - (\mu l + 3)w'h_{--}) \right] d\phi , \quad (\text{III.24})$$

with no possible central extension.

What we found in this section is somewhat reminiscent of what occurs for a scalar field in  $d$  dimensions with an arbitrary functional relation  $a_+(a_-)$ . Although the metric still has the same asymptotic AdS invariance, the scalar field breaks then the symmetry down to  $R \times SO(d-1)$  because the relationship is not maintained under the action of an asymptotic radial diffeomorphism [18]. This kind of breaking of asymptotic AdS invariance for scalar fields has been considered in [25], following ideas from the AdS/CFT correspondence [26].

## IV. CONCLUSIONS

In this paper we have presented an extended set of boundary conditions that includes the ones recently proposed in [11] with a slower decay at infinity than the one for pure standard gravity discussed in [13]. This set considers simultaneously both chiralities (below the chiral point), and it was shown that requiring well-defined charges, whose variations are not only finite but also integrable, forces one to impose a functional relationship between the two chiralities. It was proved that there is no possible relation that preserves the conformal symmetry at infinity, except setting one of the two chiralities to zero; otherwise, the best that one can achieve is  $\text{Virasoro} \times R$ . Therefore, the boundary conditions given in the earlier treatment appear to be the best possible ones compatible with conformal symmetry.

It would be interesting to explore whether a similar treatment of the asymptotic behavior for topologically massive gravity along the lines presented here could be performed around warped AdS [27], [28], [29].

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## Appendix A: Hamiltonian Formulation

### 1. Action in first order form

The canonical analysis of topologically massive gravity has been performed in [23, 30]. We shall follow very closely the procedure devised in [23] to reach the Hamiltonian form of the theory. This procedure has the advantage of bypassing many of the technical intricacies associated with a more conventional application of the canonical formalism for theories with higher order derivatives.

By introducing Lagrange multipliers for the torsion constraints and using as new connection [31]

$$A^a = \omega^a + \mu e^a, \quad (\text{A.1})$$

one can rewrite the action in first order form as follows,

$$I[e, A, \beta] = \frac{1}{\mu} I_{CS}[A] + \int [\beta^a (De_a - \mu \epsilon_{abc} e^b \wedge e^c) - \alpha \epsilon_{abc} e^a \wedge e^b \wedge e^c] \quad (\text{A.2})$$

where  $I_{CS}[A]$  reads

$$I_{CS}[A] = \int \left[ A^a \wedge \left( dA_a + \frac{1}{3} \epsilon_{abc} A^b \wedge A^c \right) \right] \quad (\text{A.3})$$

and where the covariant derivative is taken with respect to the connection  $A$ . The parameter  $\alpha$  is given by

$$\alpha = \frac{1}{3} \left( \mu^2 - \frac{1}{l^2} \right). \quad (\text{A.4})$$

The independent variables are the components of the triad  $e^a = e^a_\mu dx^\mu$ , the components of the connection  $A^a = A^a_\mu dx^\mu$  and the components of the Lagrange multiplier  $\beta^a = \beta^a_\mu dx^\mu$ .

The action (A.2) is already in the Hamiltonian form “ $\int dt(p\dot{q} - \lambda^\alpha H_\alpha(q, p))$ ” where the  $p$ ’s and the  $q$ ’s are the spatial components of the dynamical variables and where the Lagrange multipliers are their temporal components. Explicitly, one can write (A.2) as

$$I[e^a_i, A^a_i, \beta^a_i; e^a_0, A^a_0, \beta^a_0] = \int d^3x \left[ \epsilon^{ij} \left( -\frac{1}{\mu} A^a_i \dot{A}_{aj} - \beta^a_i \dot{e}_{aj} \right) - A^a_0 J_a - \beta^a_0 T_a - e^a_0 B_a \right] \quad (\text{A.5})$$

with

$$J_a = -\frac{2}{\mu}\epsilon^{ij}\left(F_{aij} + \frac{\mu}{2}\epsilon_{abc}\beta_i^b e_j^c\right) \quad (\text{A.6})$$

$$T_a = -\epsilon^{ij}\left(D_i e_{aj} - \mu\epsilon_{abc}e_i^b e_j^c\right) \quad (\text{A.7})$$

$$B_a = -\epsilon^{ij}\left(D_i\beta_{aj} - 2\mu\epsilon_{abc}\beta_i^b e_j^c - 3\alpha\epsilon_{abc}e_i^b e_j^c\right) \quad (\text{A.8})$$

and  $F_a = dA_a + \frac{1}{2}\epsilon_{abc}A^b \wedge A^c$ .

The kinetic term in the action (A.5) implies the following Poisson brackets among the variables

$$[A_i^a, A_j^b] = \frac{\mu}{2}\eta^{ab}\epsilon_{ij} \quad (\text{A.9})$$

$$[e_i^a, \beta_j^b] = \eta^{ab}\epsilon_{ij} \quad (\text{A.10})$$

## 2. Constraints

Varying the Lagrange multipliers yield the constraints  $J_a \approx 0$ ,  $T_a \approx 0$  and  $B_a \approx 0$ . Their Poisson brackets have been computed in [23]. For completeness, we reproduce them here,

$$[J[\xi], C[\eta]] = -C[\xi \times \eta] \quad (\text{A.11})$$

$$[T[\xi], T[\eta]] = -\frac{\mu}{2} \int d^2x \xi^a \eta^b (\epsilon^{ij} e_{ai} e_{bj}) \quad (\text{A.12})$$

$$[B[\xi], T[\eta]] = -\frac{\mu}{2} J[\xi \times \eta] + 2\mu T[\xi \times \eta] + \frac{\mu}{2} \int d^2x \xi^a \eta^b (\epsilon^{ij} \beta_{ai} e_{bj} - \eta_{ab} \epsilon^{ij} \beta_i^c e_{cj}) \quad (\text{A.13})$$

$$[B[\xi], B[\eta]] = 2\mu B[\xi \times \eta] + 6\alpha T[\xi \times \eta] - \frac{\mu}{2} \int d^2x \xi^a \eta^b (\epsilon^{ij} \beta_{ai} \beta_{bj}) \quad (\text{A.14})$$

Here, the notation  $C[\xi]$  stands for  $\int d^2x \xi^a C_a$  for any constraint  $C_a \equiv J_a, T_a$  or  $B_a$ , where  $\xi^a$  are arbitrary parameters assumed for the moment to have compact support in order to avoid surface terms at infinity (dealt with below). The notation  $\xi \times \eta$  is short for  $(\xi \times \eta)^a = \epsilon^{abc} \xi_b \eta_c$ . The first bracket (A.11) simply follows from the fact that the constraint  $J[\xi]$  is the generator of local Lorentz transformations.

The brackets between the constraints imply in particular the following equation for  $\dot{T}_a$

$$\dot{T}_a \approx -\frac{\mu}{2} \beta_0^b \epsilon^{ij} e_{ai} e_{bj} - \frac{\mu}{2} e_0^b \epsilon^{ij} \beta_{bi} e_{aj} + \frac{\mu}{2} e_{a0} \Delta \quad (\text{A.15})$$

where

$$\Delta = \epsilon^{ij} \beta_i^c e_{cj}. \quad (\text{A.16})$$

To analyse the nature of the constraints and determine whether they imply further constraints, it is convenient to introduce the components  $n_a$  in the triad frame of the normal to the hypersurfaces  $x^0 = \text{const}$ . These are defined through

$$n^a e_{ai} = 0, \quad n^a n_a = -1 \quad (\text{A.17})$$

( $n^a e_{ai} \equiv n_i = 0$ ). As observed in [32], the  $n^a$ 's are functions of the canonical variables  $e_{ai}$  only and do not depend on the Lagrange multipliers. Explicitly,  $n^a = u^a / \sqrt{-u^b u_b}$  with  $u^a = -\epsilon^{abc} \epsilon^{ij} e_{ai} e_{bj}$ . Note that  $n^a e_{a0} = n_0 \neq 0$ . Some useful relations are

$$e_{ai} e^a_j = g_{ij} \quad (\text{A.18})$$

$$e_b^i e_{ai} = \eta_{ab} + n_a n_b \quad (\text{A.19})$$

$$\epsilon^{ij} e_{ai} = \sqrt{g} \epsilon_{abc} e^{bj} n^c \quad (\text{A.20})$$

where  $e_b^i = g^{ij} e_{bj}$  with  $g^{ij}$  the inverse of the spatial two-dimensional metric  $g_{ij}$  and  $g = \det(g_{ij})$ . From the last relation, one derives

$$\epsilon^{ij} e_{ai} e_{bj} = \sqrt{g} \epsilon_{abc} n^c \quad (\text{A.21})$$

and

$$\epsilon^{abc} \epsilon^{ij} e_{ai} e_{bj} n_c = 2\sqrt{g} \quad (\text{A.22})$$

Projecting (A.15) along the normal yields  $n^a \dot{T}_a \approx n^a e_{a0} \Delta$ . Hence, since the constraints must be preserved in time and since  $n^a e_{a0} \neq 0$ , one gets the further constraint

$$\Delta \approx 0. \quad (\text{A.23})$$

The full set of constraints at this stage is given by  $\{J_a, T_a, B_a, \Delta\} \approx 0$ . To verify that there is no other constraint and to separate the constraints into first class and second class, it is convenient to redefine the constraint  $B_a$  as [23]

$$\hat{B}[\xi] = B[\xi] + T[\hat{\xi}] , \quad (\text{A.24})$$

where

$$\hat{\xi}^a = e^{aj} \beta_{bj} \xi^b + f n^a , \quad (\text{A.25})$$

and  $f$  is given by

$$f = -\beta_{ai} n^a e^{ci} \xi_c + \left( \beta_{ai} e^{ai} + \frac{9\alpha}{\mu} \right) n^c \xi_c . \quad (\text{A.26})$$

One easily verifies that  $e_{ai}\hat{\xi}^a = \beta_{ai}\xi^a$  and that the constraints  $\hat{B}_a \approx 0$  are first class. Our expression for  $\hat{\xi}^a$  can be checked to coincide with the one given in [23], but we have rewritten it in a way that makes it clear that the redefinition of the constraints  $B_a \rightarrow \hat{B}_a$  involves only the canonical variables *and not the Lagrange multipliers*  $e_t^a$  or  $\beta_t^a$  (*which would not be permissible*).

One can in fact verify that

$$h[\xi] = \int d^2x \left( \xi^a \hat{B}_a + A_\mu^a \xi^\mu J_a \right) \quad (\text{A.27})$$

generates on shell the Lie derivatives of the canonical variables, i.e.,

$$[X, h[\xi]] = -L_\xi X \quad (\text{A.28})$$

modulo terms that vanish when the equations of motion hold [23]. Here,  $\xi^a$  is related to the vector field  $\xi^\mu$  parametrizing the infinitesimal diffeomorphism through  $\xi^a = e_\mu^a \xi^\mu$ .

While the constraints  $\{J_a, \hat{B}_a\}$  are first class, the remaining constraints  $\{T_a, \Delta\}$  are second class. Their (invertible) brackets are explicitly computed in [23], to which we refer. There are thus 6 first class constraints and 4 second class constraints. Given that the number of canonical variables is 18, this gives  $18 - 2 \times 6 - 4 = 2$  physical canonical variables, corresponding to one conjugate canonical pair – and hence one physical degree of freedom – (per space point), independently of the value of  $\mu$ .

## Appendix B: Surface integrals

We now turn to the discussion of the surface terms that must be added to the diffeomorphism generators under the boundary conditions given in the text.

These boundary conditions have been written in terms of the metric and our first task is to rewrite them in terms of the triads since the canonical formalism has been developed in terms of these. To that end, we find it convenient to freeze the local Lorentz gauge freedom at infinity by imposing that the triads are asymptotically given by

$$e_\mu^a = \bar{e}_\mu^a + \Delta e_\mu^a \quad (\text{B.1})$$

where the  $\bar{e}_\mu^a$ 's are the following choice of AdS triads

$$\begin{aligned}\bar{e}^0 &= \left(1 + \frac{r^2}{l^2}\right)^{1/2} dt = \frac{l}{2} \left(1 + \frac{r^2}{l^2}\right)^{1/2} (dx^+ + dx^-) , \\ \bar{e}^1 &= \left(1 + \frac{r^2}{l^2}\right)^{-1/2} dr , \\ \bar{e}^2 &= r d\phi = \frac{1}{2} r (dx^+ - dx^-) ,\end{aligned}\tag{B.2}$$

and the perturbation  $\Delta e_\mu^a$  is related to the metric perturbation  $\Delta g_{\mu\nu}$  through

$$\Delta e_\mu^a = \bar{e}_\rho^a \bar{g}^{\rho\sigma} \left( \frac{1}{2} \Delta g_{\sigma\mu} - \frac{1}{8} \Delta g_{\sigma\gamma} \bar{g}^{\gamma\beta} \Delta g_{\beta\mu} \right) .\tag{B.3}$$

Note that in the case of asymptotic conditions invariant under the full conformal symmetry, for which one chirality is switched off, only the linear term in the metric perturbation is actually required.

The ordinary Lie derivative of the triads does not preserve this gauge-fixing of the triads. More precisely, when performing a diffeomorphism generated by a vector field  $\xi^\mu$  that approaches at infinity an asymptotic symmetry according to (II.2), one must compensate by a local Lorentz transformation that brings one back to that chosen gauge. Explicitly, the parameter of that compensating Lorentz transformation is found to be

$$-A_\mu^a \xi^\mu + \left( \mu \pm \frac{1}{\ell} \right) e_\mu^a \xi_{(\pm)}^\mu\tag{B.4}$$

where  $\xi_{(\pm)}^\mu$  are the pieces of  $\xi^\mu$  asymptotically determined by  $T^+$  and  $T^-$ , respectively. Hence, the full generators, containing the diffeomorphism part plus the compensating Lorentz transformation, read

$$H[\xi_{(\pm)}] = \hat{B}[\xi_{(\pm)}] + \left( \mu \pm \frac{1}{\ell} \right) J[\xi_{(\pm)}] + \text{surface terms}\tag{B.5}$$

where  $\xi_{(\pm)}^a = e_\mu^a \xi_{(\pm)}^\mu$ .

Up to the surface terms,  $H[\xi_{(\pm)}]$  coincides with the  $L_\pm[\xi]$  of reference [23], but we have followed a different logic to arrive at that expression. We can thus take over from that reference the computation of the terms that arise upon integration by parts in the bulk piece of  $H[\xi_{(\pm)}]$ . One finds

$$\delta Q_\pm = \int_{\partial\Sigma} \xi_{(\pm)}^\mu \left[ e_\mu^a \delta \beta_{a\phi} + \beta_\mu^a \delta e_{a\phi} + 2 \left( 1 \pm \frac{1}{\mu\ell} \right) e_\mu^a \delta A_{a\phi} \right] d\phi .\tag{B.6}$$



After straightforward algebra, one gets from (B.6) the expression (III.2) for  $\delta Q_{\pm}$ .

This can be seen as follows: The field equation that comes from the variation with respect to  $\beta^a_{\mu}$  implies the vanishing of the torsion, which allows to express the spin connection  $\omega^a_{\mu}$  in terms of the triad (given by Eq. (B.1), with (B.2) and (B.3)), hence determining the field  $A^a_{\mu}$  defined in (A.1). Analogously, varying with respect to  $A^a_{\mu}$  gives an algebraic equation that allows to find  $\beta^a_{\mu}$ , which is given by

$$\beta^a_{\mu} = -\frac{2}{\mu} e^a_{\nu} \left( R^{\nu}_{\mu} - \frac{1}{4} \delta^{\nu}_{\mu} R + \frac{\mu^2}{2} \delta^{\nu}_{\mu} \right) .$$

The asymptotic form of the field equation  $E_{rr} = 0$  in (C.2), which turns out to be a constraint, is also useful in order to express the terms appearing  $\delta Q_{\pm}$ , given by

$$\xi_{\pm}^{\mu} e^a_{\mu} \delta \beta_{a\phi} = \pm \frac{1}{4\mu l^2} (\mu^2 l^2 - 1) T^{\pm} [2\delta h_{\pm\pm} r^{1\pm\mu l} + 2\delta f_{+-} - 2\delta f_{\pm\pm} + 11h_{\pm\pm} \delta h_{\mp\mp} + 9h_{\mp\mp} \delta h_{\pm\pm}]$$

$$\xi_{\pm}^{\mu} \beta^a_{\mu} \delta e_{a\phi} = \pm \frac{1}{4\mu l^2} (\mu^2 l^2 - 1) T^{\pm} [-2\delta h_{\pm\pm} r^{1\pm\mu l} + 2\delta f_{+-} - 2\delta f_{\pm\pm} + 3h_{\pm\pm} \delta h_{\mp\mp} + h_{\mp\mp} \delta h_{\pm\pm}]$$

$$\xi_{\pm}^{\mu} e^a_{\mu} \delta A_{a\phi} = \frac{1}{4l} T^{\pm} [2(1 \pm \mu l) \delta f_{\pm\pm} + (1 \mp \mu l) [2\delta f_{+-} + (4 \pm \mu l) \delta(h_{++} h_{--})]]$$

Therefore, the divergences coming from the first and second terms above cancel out, and the variation of the charge in (B.6) reduces to

$$\delta Q_{\pm} = \left( 1 \pm \frac{1}{\mu l} \right) \delta Q_{\pm}^0 + \delta Q_{\pm}^{nl} ,$$

whre  $\delta Q_{\pm}^0$  and  $\delta Q_{\pm}^{nl}$ , are given in Eqs. (III.3) and (III.4), respectively.

### Appendix C: Asymptotic form of the field equations

Once the auxiliary fields  $A^a_{\mu}$  and  $\beta^a_{\mu}$  are eliminated, the field equations coming from (I.1) read

$$E_{\mu\nu} := G^{\mu}_{\sigma} - \frac{1}{l^2} \delta^{\mu}_{\sigma} - \frac{1}{\mu} C^{\mu}_{\sigma} = 0 , \quad (C.1)$$

where  $\mu \neq 0$  is the mass parameter,  $l$  is the AdS radius, and  $C^{\mu}_{\sigma} := (-g)^{-1/2} \epsilon^{\mu\nu\rho} \nabla_{\nu} (R_{\rho\sigma} - \frac{1}{4} g_{\rho\sigma} R)$ , stands for the Cotton tensor.

In the case of  $|\mu l| < 1$ , for the asymptotic conditions given by (II.1), the leading terms of the relevant field equations reduce to

$$E_{rr} = r^{-4} [-l^{-2} f_{rr} + 4f_{+-} + (5 - \mu^2 l^2) h_{++} h_{--}] + \dots \quad (\text{C.2})$$

$$E_{r\pm} = \frac{1}{2} \left( 1 \pm \frac{1}{\mu l} \right) r^{-3} [4\partial_{\mp} f_{\pm\pm} - (1 \mp \mu l) [(1 \mp \mu l) h_{\mp\mp} \partial_{\pm} h_{\pm\pm} + (3 \mp \mu l) h_{\pm\pm} \partial_{\pm} h_{\mp\mp}]] + \dots \quad (\text{C.3})$$

where the first equation was used to simplify the second one. The remaining field equations are of subleading orders as compared with the asymptotic behaviour of the metric.

Note that, by integrating the equations from infinity, there is no apparent obstruction coming from (C.2) and (C.3) in order to construct solutions where both chiralities are present. Indeed, one could choose  $h_{++}$  and  $h_{--}$  to be independent functions of  $x^+$  and  $x^-$ ; however, as explained in Section III A, integrability of the surface generators requires them to be functionally dependent. For a generic choice of  $h_{++}(h_{--})$ , the asymptotic symmetry is broken down to  $R \times U(1)$ , while if one of the relations (III.9) or (III.11) are fulfilled, then the symmetry enhances to  $\text{Virasoro} \times R$ . Full conformal symmetry at infinity is recovered when a single chirality is switched on. In this case, there is no obstruction for solutions generalizing the pp-waves to exist ( $h_{++}$  or  $h_{--}$  depend explicitly on  $x^+$  and  $x^-$ ). And indeed, this is realized for the class of metrics recently found in [4], [34], [5].

## Appendix D: Asymptotic symmetries and canonical generators algebra

### 1. Full conformal symmetry

When one chirality is switched off (i.e., when  $h_{++}$  or  $h_{--}$  vanishes), as explained in Section III C, the asymptotic symmetries correspond to the conformal group in two dimensions generated by (II.2). Then,  $f_{++}$  and  $f_{--}$  are straightforwardly found to transform as

$$\delta_{\eta} f_{++} = 2f_{++} \partial_+ T^+ + T^- \partial_- f_{++} + T^+ \partial_+ f_{++} - l^2 (\partial_+ T^+ + \partial_+^3 T^+)/2, \quad (\text{D.1})$$

$$\delta_{\eta} f_{--} = 2f_{--} \partial_- T^- + T^- \partial_- f_{--} + T^+ \partial_+ f_{--} - l^2 (\partial_- T^- + \partial_-^3 T^-)/2. \quad (\text{D.2})$$

From Eq. (C.3), one verifies that, on shell

$$\partial_+ f_{--} = 0 = \partial_- f_{++}, \quad (\text{D.3})$$

and so (D.1) and (D.2) reduce to

$$\delta_\eta f_{++} = 2f_{++}\partial_+ T^+ + T^+\partial_+ f_{++} - l^2 (\partial_+ T^+ + \partial_+^3 T^+)/2 , \quad (\text{D.4})$$

$$\delta_\eta f_{--} = 2f_{--}\partial_- T^- + T^-\partial_- f_{--} - l^2 (\partial_- T^- + \partial_-^3 T^-)/2 . \quad (\text{D.5})$$

Therefore, the variation of the charges (III.12) reads

$$\begin{aligned} \delta_{\eta_2} Q_\pm [T_1^\pm] &= [Q_\pm [T_1^\pm], Q_+ [T_2^+] + Q_- [T_2^-]] \\ &= \frac{2}{l} \left( 1 \pm \frac{1}{\mu l} \right) \int T_1^\pm \delta_{\eta_2} f_{\pm\pm} d\phi , \\ &= \frac{2}{l} \left( 1 \pm \frac{1}{\mu l} \right) \int d\phi \left[ (T_1^\pm \partial_\pm T_2^\pm - T_2^\pm \partial_\pm T_1^\pm) f_{\pm\pm} - \frac{l^2}{2} T_1^\pm (\partial_\pm T_2^\pm + \partial_\pm^3 T_2^\pm) \right] \\ &= Q_\pm [[T_1^\pm, T_2^\pm]] - \left( 1 \pm \frac{1}{\mu l} \right) l \int d\phi T_1^\pm (\partial_\pm T_2^\pm + \partial_\pm^3 T_2^\pm) , \end{aligned}$$

which implies that  $Q_+$  and  $Q_-$  commute with each other and each fulfills the Virasoro algebra with central charges

$$c_\pm = \left( 1 \pm \frac{1}{\mu l} \right) c \quad (\text{D.6})$$

(see [24] for general theorems).

## 2. Two chiralities present

In the case of  $|\mu l| < 1$ , one obtains quite generally that  $f_{++}$  and  $f_{--}$  respectively transform as in Eqs. (D.1) and (D.2) under the action of the Virasoro algebra, and

$$\begin{aligned} \delta_\eta h_{++} &= \frac{1}{2} [(3 - \mu l) \partial_+ T^+ - (\mu l + 1) \partial_- T^-] h_{++} + T^- \partial_- h_{++} + T^+ \partial_+ h_{++} , \\ \delta_\eta h_{--} &= \frac{1}{2} [(\mu l - 1) \partial_+ T^+ + (3 + \mu l) \partial_- T^-] h_{--} + T^- \partial_- h_{--} + T^+ \partial_+ h_{--} . \end{aligned}$$

As we have seen, when both chiralities are present, only one copy of the Virasoro algebra actually survives as asymptotic symmetry. Let us assume first that we adopt the relation (III.9) between  $h_{++}$  and  $h_{--}$ . Thus, from Eq. (III.16) one obtains

$$\delta_{\xi_2} Q_+ [\xi_1] = \frac{2}{l} \left( 1 + \frac{1}{\mu l} \right) \int T_1^+ \delta_{\xi_2} f_{++} d\phi .$$

Remarkably, by virtue of (III.9), the nonlinear terms of the field Eq.  $E_{r+} = 0$  in (C.3) vanish, so that it reduces to

$$\partial_- f_{++} = 0 ,$$

and hence  $Q_+$  fulfills the Virasoro algebra with the corresponding central charge  $c_+$ .

In the case of  $Q_-$ , from (III.18) one obtains

$$\delta_{\xi_2} Q_-[\xi_1] = \frac{2}{l} \left( 1 - \frac{1}{\mu l} \right) \int T_1^- \left[ \delta_{\xi_2} f_{--} - 2a_0^+ \frac{\mu l + 1}{\mu l - 1} (h_{--})^{\frac{3-\mu l}{\mu l - 1}} \delta_{\xi_2} h_{--} \right] d\phi ,$$

where  $T_1^-$  and  $T_2^-$  are constants. Since  $\delta_{\xi_2} f_{--}$  and  $\delta_{\xi_2} h_{--}$  can be read from (D.1) and (III.6), respectively, the variation of the charge reads

$$\delta_{\xi_2} Q_-[\xi_1] = \frac{2}{l} \left( 1 + \frac{1}{\mu l} \right) \int d\phi T_1^- T_2^+ \partial_+ f_{--} . \quad (\text{D.7})$$

In this case, once one uses the relation (III.9), the nonlinear terms of the field Eq.  $E_{r-} = 0$  in (C.3) do not vanish, so that it reads

$$\partial_+ f_{--} = 2a_0^+ \frac{\mu l + 1}{\mu l - 1} (h_{--})^{\frac{3-\mu l}{\mu l - 1}} \partial_- h_{--} .$$

Thus, plugging this field equations into (D.7) and integrating by parts, one obtains that

$$\delta_{\xi_2} Q_-[\xi_1] = 0 .$$

It is simple to verify that for the other condition in Eq. (III.11), the corresponding charge (III.19) generates a Virasoro algebra with central extension  $c_-$  and commutes with the abelian one in (III.21). Indeed, this corresponds to making  $\mu \longleftrightarrow -\mu$  and  $x^+ \longleftrightarrow x^-$  in the computations of this section.

Analogously, it is very simple to treat the case of an arbitrary relation between  $h_{++}$  and  $h_{--}$ . Since the conditions (III.8), (III.10) are fulfilled only for the zero modes of both copies of the Virasoro symmetry, the algebra of the canonical generators coincides then with the one of the remaining asymptotic symmetries,  $R \times U(1)$ , with no central extension.

## Appendix E: Chiral point

For completeness, we derive below the formulas relevant to the chiral point  $|\mu l| = 1$ . This is somewhat out of the main line of our paper which explores the possibility (only available for  $|\mu l| < 1$ ) of switching on simultaneously both chiralities. But since these formulas, announced in [11], are easily derived from the above computations, we provide them here. We assume  $\mu l = 1$ . The case  $\mu l = -1$  just corresponds to the interchange  $x^+ \longleftrightarrow x^-$ .

## 1. Surface integrals

In the case  $\mu l = 1$ , only the negative chirality is present (i.e.,  $h_{++} = h_{r+} = 0$ ). The suitable asymptotic behaviour for  $\Delta g_{\mu\nu}$  possessing full conformal invariance at infinity is given by [11]

$$\begin{aligned}
\Delta g_{rr} &= f_{rr} r^{-4} + \dots \\
\Delta g_{r+} &= f_{r+} r^{-3} + \dots \\
\Delta g_{r-} &= \tilde{h}_{r-} r^{-3} \log(r) + \tilde{f}_{r-} r^{-3} + \dots \\
\Delta g_{++} &= f_{++} + \dots \\
\Delta g_{+-} &= f_{+-} + \dots \\
\Delta g_{--} &= \tilde{h}_{--} \log(r) + \tilde{f}_{--} + \dots
\end{aligned} \tag{E.1}$$

which accommodates solutions of the form (I.4) having constant curvature at the asymptotic region. The variation of the charges is then obtained following the same procedure as explained above for the asymptotic behavior of the metric in (E.1). The only nonvanishing terms in (B.6) are then given by

$$\xi_-^\mu e_\mu^a \delta \beta_{a\phi} = \frac{2}{l} T^- \delta \tilde{h}_{--}$$

$$\xi_+^\mu e_\mu^a \delta A_{a\phi} = \frac{1}{l} T^+ \delta f_{++}$$

so that the variation of the charges reads [11]

$$\begin{aligned}
\delta Q_\pm &= \frac{4}{l} \int T^\pm \delta f_{++} d\phi, \\
\delta Q_- &= \frac{2}{l} \int T^- \delta \tilde{h}_{--} d\phi.
\end{aligned}$$

The charges are then given by

$$Q_+ = \frac{4}{l} \int T^+ f_{++} d\phi, \text{ and } Q_- = \frac{2}{l} \int T^- \tilde{h}_{--} d\phi. \tag{E.2}$$

Note that  $Q_-[T^-]$  does not vanish identically, but rather, the relaxation term  $\tilde{h}_{--}$  does contribute to it. This behavior is somehow similar to what occurs for scalar fields that saturates the BF bound [17].

Note also that the Virasoro generators with both chiralities are non zero at the chiral point (while one chiral set of them does vanish under the boundary conditions of [13]). The

asymptotic form of the metric (E.1) agrees with the one in [33] and it can be obtained from (II.1) in the limit  $\mu l \rightarrow 1$ . This can be seen as follows: Requiring the curvature to be constant at infinity implies that the branch with positive chirality in (II.1) has to be switched off (i.e., one makes  $h_{++} = 0$ , so that  $h_{r+}$  can be gauged away). Then, in the limit  $\mu l \rightarrow 1$ , since  $r^{1-\mu l} = 1 + (1 - \mu l) \log(r) + \dots$ , one recovers (E.1) with [35]

$$\begin{aligned}\tilde{h}_{--} &= (1 - \mu l) h_{--} \quad ; \quad \tilde{f}_{--} = h_{--} + f_{--} \\ \tilde{h}_{r-} &= (1 - \mu l) h_{r-} \quad ; \quad \tilde{f}_{r-} = h_{r-} + f_{r-}\end{aligned}\tag{E.3}$$

It is amusing to verify that the charges at the chiral point in Eq. (E.2) can be obtained from the ones off the chiral point in (III.12) in the limit  $\mu l \rightarrow 1$ .

## 2. Integration from infinity

The field equations can easily be integrated from infinity in the chiral case. For  $\mu l = 1$  and the asymptotic conditions given by (E.1), the leading terms of the relevant field equations reduce to

$$E_{rr} = r^{-4} [-l^{-2} f_{rr} + 4f_{+-}] + \dots\tag{E.4}$$

$$E_{r+} = r^{-3} [4\partial_- f_{++}] + \dots\tag{E.5}$$

$$E_{r-} = r^{-3} [2\partial_+ \tilde{h}_{--}] + \dots\tag{E.6}$$

where the first equation was used to simplify the second and the third ones, and the remaining field equations are of subleading orders as compared with the asymptotic behaviour of the metric.

## 3. Central charges

We now turn to the computation of the central charges.

In the case of  $\mu l = 1$ , under the action of the Virasoro symmetry (II.2), one obtains the same Eq. (D.1), and

$$\delta_\eta h_{--} = 2h_{--}\partial_- T^- + T^- \partial_- h_{--} + T^+ \partial_+ h_{--} \ ,$$

and the asymptotic field equations (E.5), (E.6) read

$$\partial_- f_{++} = 0, \text{ and } \partial_+ h_{--} = 0.$$

Note that this time the equations do not impose  $\partial_+ f_{--} = 0$  and furthermore, the transformation rule of  $f_{--}$  also differs from the one found off the chiral point. The variation of the charges (E.2) then reads

$$\begin{aligned}
\delta_{\eta_2} Q_+[T_1^+] &= [Q_+[T_1^+], Q_+[T_2^+] + Q_-[T_2^-]] \\
&= \frac{4}{l} \int T_1^+ \delta_{\eta_2} f_{++} d\phi , \\
&= \frac{4}{l} \int d\phi \left[ (T_1^+ \partial_+ T_2^+ - T_2^+ \partial_+ T_1^+) f_{++} - \frac{l^2}{2} T_1^+ (\partial_+ T_2^+ + \partial_+^3 T_2^+) \right] \\
&= Q_+ [[T_1^+, T_2^+]] - 2l \int d\phi T_1^+ (\partial_+ T_2^+ + \partial_+^3 T_2^+) ,
\end{aligned}$$

and

$$\begin{aligned}
\delta_{\eta_2} Q_-[T_1^-] &= [Q_-[T_1^-], Q_+[T_2^+] + Q_-[T_2^-]] \\
&= \frac{2}{l} \int T_1^- \delta_{\eta_2} h_{--} d\phi , \\
&= \frac{2}{l} \int d\phi (T_1^- \partial_- T_2^- - T_2^- \partial_- T_1^-) h_{--} \\
&= Q_- [[T_1^-, T_2^-]] .
\end{aligned}$$

Therefore, one finds that both  $Q_+[T^+]$  and  $Q_-[T^-]$  fulfill the Virasoro algebra with the central charges

$$c_+ = 2c, \quad c_- = 0. \quad (\text{E.7})$$

The central charge  $c_-$  vanishes because the inhomogeneous terms  $-l^2 (\partial_- T^- + \partial_-^3 T^-) / 2$  are absent from  $\delta_{\eta} \tilde{h}_{--}$ .

The vanishing of the central charge  $c_-$  is somehow puzzling from the point of view of conformal field theory. Progress in this direction has been recently achieved in Refs. [36], [37], [38], [39].

The results of this section have been confirmed following different approaches in Refs. [35], [40], and [41] (notice that [35] and [41] consider also the non-chiral point with only one chirality).

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